Notes on ME525 Applied Acoustics Lecture 7, Winter 2022

Complex Intensity, Active and Reactive Intensity

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1) The Laplacian

The Laplacian

The following is review, but it's worth having another look at the wave equation in spherical coordinates (Fig. 1). We write the wave equation (here for pressure p) without regard to coordinate system as

$$\nabla^2 p - \frac{1}{c^2} p_{tt} = 0 \tag{1}$$

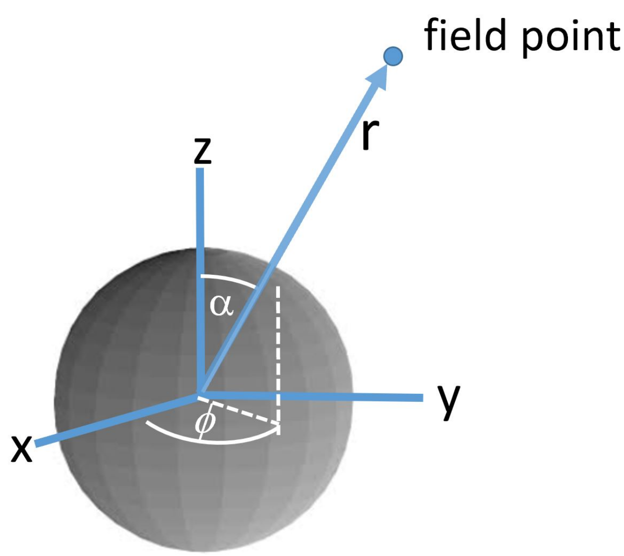

Figure 1: A spherical coordinate system centered on a sphere of arbitrary radius. Conversion to rectangular coordinates gives $x = r \sin\alpha \cos\phi$, $y = r \sin\alpha \sin\phi$, and $z = r\cos\alpha$.

And if a spherical coordinate system is being used, the big work involves the Laplacian operator in spherical coordinates (r, ϕ, α) which is

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \frac{\partial}{\partial r}) + \frac{1}{r^2 \sin^2\alpha}\frac{\partial^2}{\partial \phi^2} + \frac{1}{r^2 \sin\alpha}\frac{\partial}{\partial \alpha}(\sin\alpha \frac{\partial}{\partial \alpha}) \tag{2}$$

that operates on pressure, p. If we hypothesize that there is no variation in α or ϕ directions, or spherical symmetry, only the first term is relevant.

However its important to have a firm understanding that the wave equation can be expressed in different coordinate systems and the key feature of any coordinate system is the Laplacian differential operator ∇^2 and whether or not simplifications such as spherical symmetry can be applied. As a reminder, the Laplacian in rectangular coordinates is $\nabla^2 = \frac{\partial^2}{x^2} + \frac{\partial^2}{y^2} + \frac{\partial^2}{z^2}$. The Laplacian in these spherical, rectangular and cylindrical coordinate system is summarized in a memo on the course web site's resource page.

Complex Intensity, Active and Reactive Intensity

Let's get back red and blue "envelope" curves that I sketched for the Jacobsen data in Lecture 6 and shown here in Fig. 1. Recall that the Jacobsen data for at $kr << 1$ represents reactive intensity, and at $kr >> 1$ the situation is characterized by active intensity. These curves emerge through the concept or *complex intensity* $\vec{I}_c = \frac{1}{2} p\vec{u}^{\star}$ which was first formulated by Heyser (1986), and is discussed further in Fahy (1995) and Jacobsen and Juhl (2015).

This is best first demonstrated by a model. For example, use our standard model the pressure from a spherical wave $p(r,t) = \frac{A}{r} e^{ikr - i\omega t}$, and form $p\vec{u}^{\star}$, using in this case only a radial component u_r for $\vec{u}$. This yields

$$I_c = \frac{|A|^2}{2\rho_0 c} \left(\frac{1}{r^2} - i\frac{1}{r^2 kr}\right). \tag{3}$$

where in this case, I_c has only one component in the radial direction as in u_r.

The real and imaginary parts of I_c identify *active intensity* as $I = \mathrm{Re}\{I_c\}$, and the *reactive intensity* as $Q = \mathrm{Im}\{I_c\}$ (and in general case these are vectors $\vec{I}$ and $\vec{Q}$). Thus according to this model active intensity is

$$I = \frac{|A|^2}{2\rho_0 c r^2} \tag{4}$$

Notice: in this model active intensity is no longer time-varying but does maintain an $r-$ dependence, consistent with spherical spreading. The corresponding Umov vector for this same model for pressure (see Lecture 6)=

$$S_r(r,t) = \frac{|A|^2}{r^2 \rho_0 c} \{\cos^2(kr - \omega t + \phi_A) - \frac{\cos(kr - \omega t + \phi_A)\sin(kr - \omega t + \phi_A)}{kr}\} \tag{5}$$

Observe: the time average of $S_r(r,t)$ identified formally as $\langle S_r(r,t)\rangle$ also yields the result for I in Eq.(4). Thus, henceforth associate *active intensity* $\vec{I}$ as a time average, in some reasonable sense, of the Umov vector.

With this particular model the result $\langle S_r(r,t)\rangle$ definitely no longer has time variation, only the spatial variation by way of range r. With real data there can be slowly-varying changes, for example, going back to the Jacobsen data, Fig. 2 of Lecture 6 for case $kr >> 1$, observe the red line (a rough sketch we added to the data) is describing in some sense a kind of "running average" of the

Umov vector. (In hindsight the sketch is not so great. It really should be a bit lower.)

The imaginary part, reactive intensity, is more subtle. You get introduced to it in this class (most classes in acoustics you would not), maybe we look at a bit, but then we got to move on (those who do research with get to know it much better!) Evidently for this model Q equals $-\frac{|A|^2}{2\rho_0 c r^2 kr}$, which also no longer has time variation. Note this subtle point: the sign of reactive intensity is not of physical significance, and depends on which convention $e^{\pm i\omega t}$ is used.

How does this running average idea work with Q? Interpret reactive intensity as a running average of the envelope of reactive intensity (to the extent it exists). For example, take Fig. 1 of Lecture 6 for case $kr << 1$ and observe the blue line (again our sketch) tracing the envelope of the Umov vector, which is seen upon inspection to have a near-zero time average.

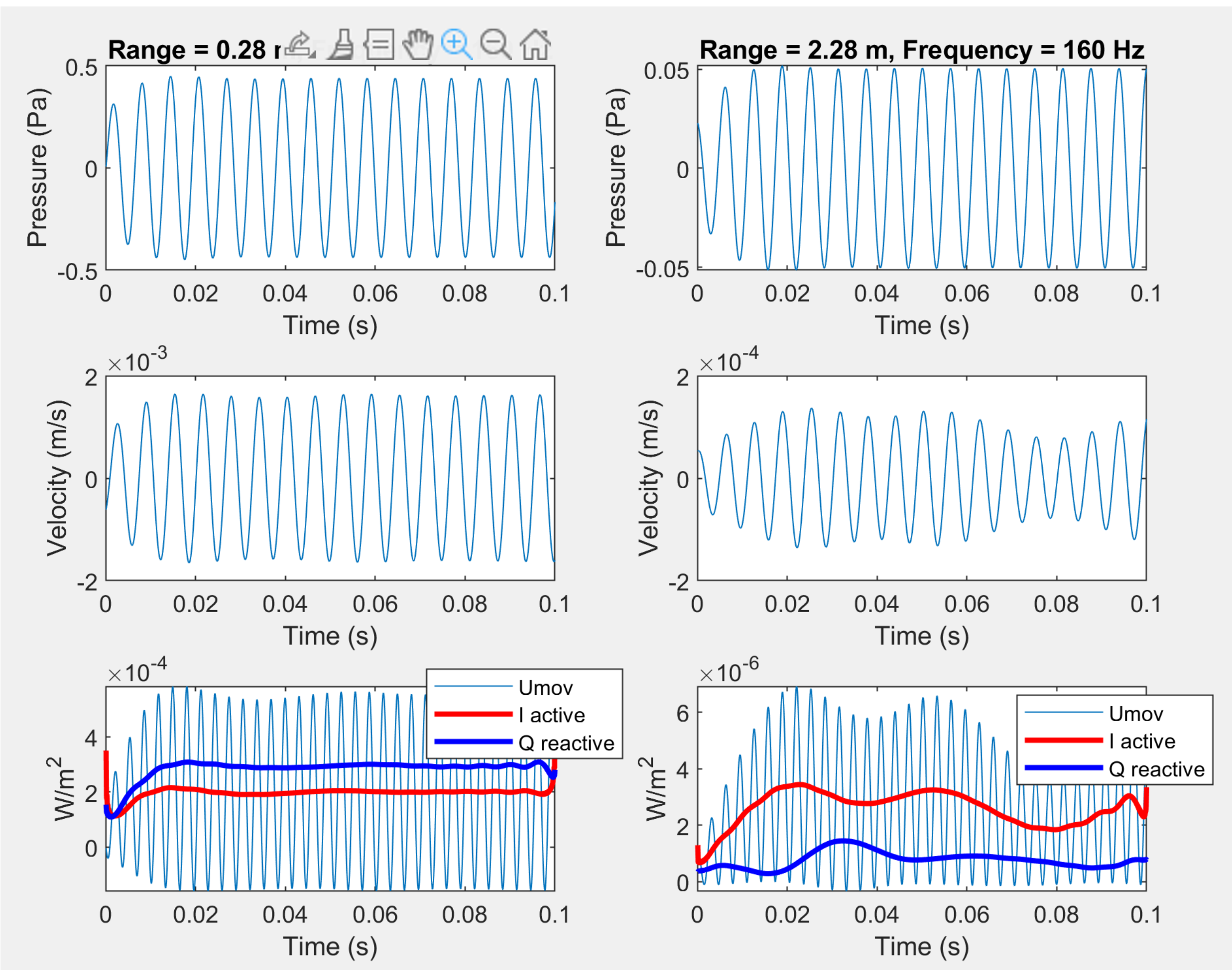

Figure 2: Acoustic pressure (top) acoustic velocity (middle) and intensity measures (bottom) based on measurements at 160 Hz, at two ranges completed at the Army Research Laboratory anachoic chamber, Dall'Osto and Dahl

A better demonstration (without sketches) is one based on our own measurements at the Army Research Laboratory's anachoic chamber made by my colleague Dr. David Dall'Osto, and me. Besides testing the instrument we were developing at the time, another strong motivation was for us was to attempt to duplicate the Jacobsen data. Shown are results from two ranges, 0.28 m and 2.28 m, from a speaker source transmitting at frequency 160 Hz, with a typical conditions for air of c = 343 m/s and ρ_0 = 1.2 kg/m^3.

At range 0.28 m, $kr \sim 0.8$, or not quite $kr << 1$, but clearly not $kr >> 1$. We anticipate a mixture of active and reactive intensity. This is suggested by inspecting the pressure and velocity time series, for which you can see a small shift in phase between pressure and velocity–or they don't quite line up. Taking a pure average $\langle S_r(r,t)\rangle$ over this 0.1 s time period, yields one value = 1.96 10^{-4} W/m^2, and the active intensity I (red line) approximately captures this number, though varies somewhat at the start. The reactive intensity Q (blue line) is mixed in with the active component and evidently higher strength or value. At range 2.28 m $kr \sim 6$, notice that oscillations in pressure and velocity align much, though not perfectly, and we can anticipate the observation that Q will have diminished considerably relative to I.

How do we find the complex intensity, real (red line) and imaginary (blue line) parts when working with this type of real-valued measurement data? In matlab a simple solution is to form the *Hilbert transform pair* of the data using, `v_complex = hilbert(v);` where `v` is the matlab variable representing a time series of velocity, and `v_complex` is the Hilbert transformed pair result. Recover a conjugate form of the velocity in matlab using `conj(v_complex)`, or recover the original real-valued time series using `real(v_complex).` This is for your background only. We use the Hilbert transform extensively in our research, but it will not be needed for homework assignments.

That said, let's press on with one more demonstration of this interesting data at the $kr \sim 0.8$ range, to further understand what is meant by active and reactive intensity. Let us split apart the acoustic velocity into two parts: v_p which represent a velocity that is exactly in-phase with pressure p, and v_q which represent a velocity that is exactly 90° out of phase with pressure.

$$v_p = \frac{\langle pv\rangle p}{\langle p^2\rangle} \tag{6}$$

where p and v are the pressure and velocity shown in Fig. 2 for the case $kr \sim 0.8$, and $\langle\rangle$ represents a time average of the 0.1 s duration shown in the figure. Then find $v_q = v - v_p$. (This amazingly simple algorithm is discussed in Stanzial *et al.* 2012.)

Notice that $\langle pv\rangle$ is the same as the time-average of the Umov vector since p and v are real-measured quantities. Let see how active and reactive intensity play out when using v_p and v_q instead of the total velocity v. Pay attention first to the left column of Fig. 3: upper is plot of p and v and they are not quite aligned in phase, which we already know given the presence of reactive intensity in Fig. 2; middle compares p and v_p, which are now perfectly aligned, and bottom compares p and v_q which are not exactly 90° out of phase.

The right column of Fig. 3 shows the corresponding Umov vectors: upper is pv (same as in

Fig. 2), middle is pv_p and lower is pv_q. The red and blue lines are active and reactive intensities, respectively. In this analysis we are able to parse them out more clearly. For example, the red active intensity line for case pv_p is precisely the same as that using the total velocity pv, but using pv_p it is more easy to see how the red line cuts through an average value of the instantaneous intensity. For case pv_p there is also a complete absence of reactive intensity.

Likewise, the blue reactive intensity line for case pv_q is precisely the same as that using the total velocity pv, but with pv_q it is more easy to see how the blue line describes an envelope of the instantaneous intensity which will otherwise have a time average equal to 0. For case pv_q there is also a complete absence of active intensity.

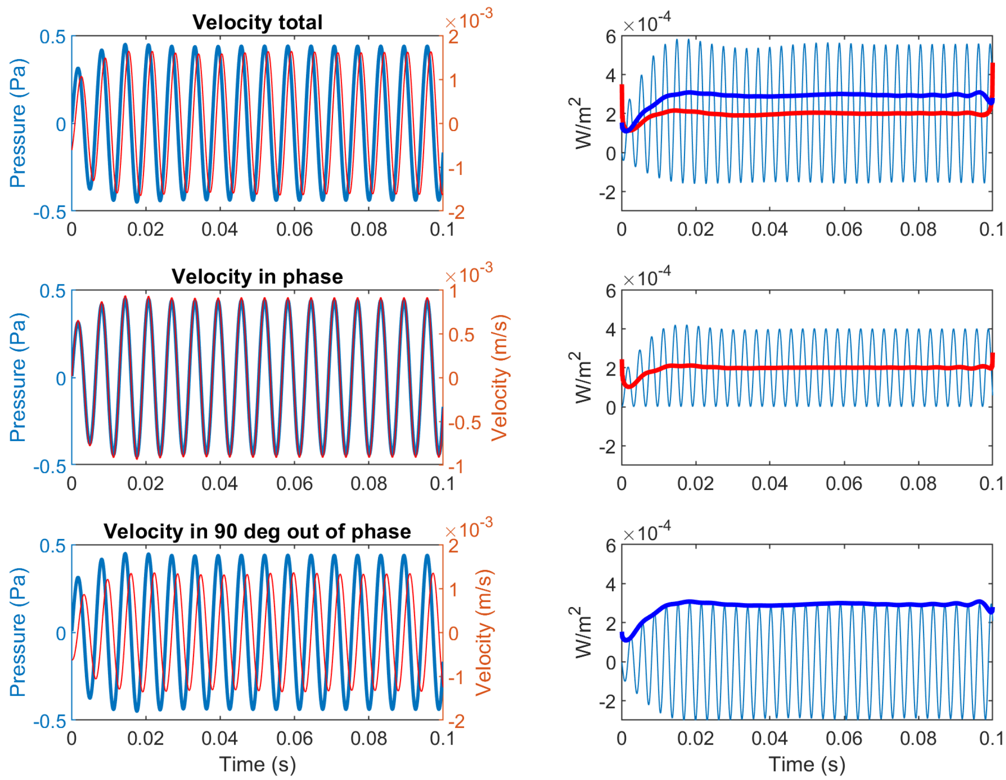

Figure 3: Further analysis of the data at $kr \sim 0.8$ based on splitting the total velocity v into in-phase component v_p and out-of-phase component v_q. Left column: comparison of pressure and velocity components. Right column: corresponding Umov vector and active (red) and reactive (blue) intensities. See text for additional discussion.

The take-home message:

- When acoustic pressure and velocity are 90° out of phase, as in the Jacobsen data for $kr << 1$ there exists reactive intensity, $\langle S_r(r,t)\rangle \sim 0$, and reactive intensity Q will describe the envelope of $S_r(r,t)$

- When acoustic pressure and velocity in phase, as in the Jacobsen data for $kr >> 1$ there exists active intensity I, $\langle S_r(r,t)\rangle$ is non-zero,

the above given in term of a single radial component, but in general there is $\vec{S}$, $\vec{I}$ and $\vec{Q}$.

We have now encountered multiple definitions relating to word intensity, all of which should have as their basic dimension Watts/m^2, or J/sec/m^2. Intensity is in general a vector quantity for which the following forms have been introduced

- Umov vector $\vec{S}$
$$\vec{S}(r,t) = Re\{p(r,t)\}Re\{\vec{u}(r,t)\} \tag{7}$$

- Complex intensity $\vec{I_c}$
$$\vec{I_c} = \frac{1}{2}p(r,t)\vec{u}^{\star}(r,t) \tag{8}$$

- *active* intensity $\vec{I} = \mathrm{Re}\{\vec{I_c}\}$ and *reactive* intensity $\vec{Q} = \mathrm{Im}\{\vec{I_c}\}$

- *plane wave* intensity $\frac{p_{rms}^2}{\rho_0 c}$ This is sometimes referred to as plane wave intensity as it is precisely the intensity one finds from a plane wave. The expression is handy to use with real data–but be careful with it usage (let's examine in a small homework problem)

References

R. B. Heyser, "Instantaneous Intensity" *Proceedings of the 81st Convention of the Audio Engineering Society* preprint 2399 (1986)
F. Jacobsen and P. M. Juhl,*Fundamentals of General Linear Acoustics* (John Wiley & Sons, 2013).
F. Fahy, *Foundations of Engineering Acoustics* (Elsevier Academic Press, San Diego, CA, 2001)
D. Stanzial, G. Sacchi and G. Schiffrer, "On the physical meaning of the power factor in acoustics,"*J. Acoust. Soc. Am.* 131, 2012.

ME525 Applied Acoustics Lecture 8, Winter 2022 Radiated acoustic power from spherical source, the $ka << 1$ limit, point source and Green's function

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1) Quick note on plane wave intensity

Using $p(r,t) = \frac{A}{r}e^{ikr-i\omega t}$ for the complex representation of a spherically symmetric pressure wave, we find by inspection the RMS pressure

$$p_{rms} = \frac{1}{\sqrt{2}}\frac{|A|}{r} \tag{1}$$

and thus conclude that a time-average of the Umov vector $\langle S \rangle$ for this case (e.g. discussed in Eq. (5) of Lecture 7 or Eq.(7) Lecture 6)

$$\langle S \rangle = \frac{p_{rms}^2}{\rho_0 c} \tag{2}$$

This is a measure commonly estimated with real data, and it applies generally to harmonic waves (e.g., a single frequency), but also is sometime applied transient sounds as in explosive waveform (multiple frequency content), and ambient noise. The intensity metric $\frac{p_{rms}^2}{\rho_0 c}$ is referred to as *plane wave* intensity because it is formally the intensity from a plane wave. One must apply some caution because measuring the pressure and forming p_{rms}^2 and dividing by $\rho_0 c$ does not generally create the vector quantity required, for example, which can be used to find total radiated acoustic power, Π from an acoustic source.

To obtain this, the average rate at which energy flows through a closed spherical surface of radius r that surrounds the source, a control surface S_c, is computed as follows

$$\Pi = \int_{S_c} \langle S \rangle \cdot d\vec{s} \tag{3}$$

So in general the dot product of the component of $\langle S \rangle$ normal to the differential area $d\vec{s}$ is computed and summed or integrated over the surface S_c. However let's say we go back to the spherically symmetric pressure field (pretty good model in many cases) the integral is then done by inspection, yielding

$$\Pi = 2\pi\frac{|A|^2}{\rho_0 c} \tag{4}$$

We see that the average rate of energy flow through any control surface surrounding the source, or the acoustic power Π, is independent of the radius of that control surface, which is consistent with conservation of energy in a lossless medium. In practice, sound absorption can reduce the total Π

(Kinsler, *et al.* 1982.) Apply the result found earlier for a spherical source of radius a, wavenumber k, $\rho_0 c$ and radial velocity amplitude u_0 at the surface of the sphere, where the complex amplitude A is derived (see Eq.(12) of Lecture 4, and Eq.(7) below), and find

$$\Pi = 2\pi a^2 |u_0|^2 \rho_0 c \frac{(ka)^2}{1+(ka)^2} \tag{5}$$

It should be more obvious now that effective radiation of acoustic power for a small source as characterized by $ka << 1$ is more difficult (think of combination of small earpod and low frequency versus high frequency sounds) as the small ka limit shows that $\Pi \sim (ka)^2$.

The spherically symmetric source in $ka << 1$ limit, and the monopole source

Continuing with this spherical wave of the form

$$p(r,t) = \frac{A}{r} e^{ikr} e^{-i\omega t} \tag{6}$$

with boundary condition $u_r(r=a) = u_0 e^{-i\omega t}$, the constant A is given by

$$A = \rho_0 c \; u_0 a (\frac{ka}{ka+i}) e^{-ika}. \tag{7}$$

Study the factor in parenthesis in the limit of $ka << 1$, find $A \approx -k\rho_0 c u_0 a^2$, based on $\frac{ka}{ka+i} = -ika$ plus order $(ka)^2$. With minor rearrangement the pressure can now be expressed as

$$p(r,t) = -i\omega(\rho_0 u_0 4\pi a^2) \frac{e^{ikr}}{4\pi r} e^{-i\omega t} \tag{8}$$

Note the $-i\omega$ (time derivative) and the $\rho_0 u_0 4\pi a^2$ (a mass) corresponding to a mass flow of (dimension M/T). Thus the strength of this acoustic source is defined by the time derivative of *mass flow*, or described another way, it is the *rate of change of mass flow* introduced per unit volume.

Next bundle everything by putting $q = -i\omega(\rho_0 u_0 4\pi a^2)$ which we shall call an *effective source strength*. Thus

$$p(r,t) = \frac{q}{4\pi r} e^{ikr - i\omega t} \tag{9}$$

where the source is at the center of the coordinate system and pressure is function only of radial coordinate r.

A further idealization is made as follows: consider the hypothetical case of a becoming progressively smaller while u_0 becomes larger such that q remains constant. This is the concept of a point source or *acoustic monopole* (Pierce, 1989), for which the source is idealized to originate from a single point. The idealization is required to confine the source within an infinitely small space, or single point, however in practice any small source with time-varying mass of fluid in any small volume enclosing the source has all the attributes of a point source (Pierce, 1989).

The Green's function

We further generalize things to find the pressure at a *field point* $\vec{r}$, given a source at an arbitrary *source point* $\vec{r_0}$ that need not be at origin (Fig. 1) as follows:

$$p(r,t) = \frac{q}{4\pi|\vec{r}-\vec{r_0}|}e^{ik|\vec{r}-\vec{r_0}|-i\omega t} \tag{10}$$

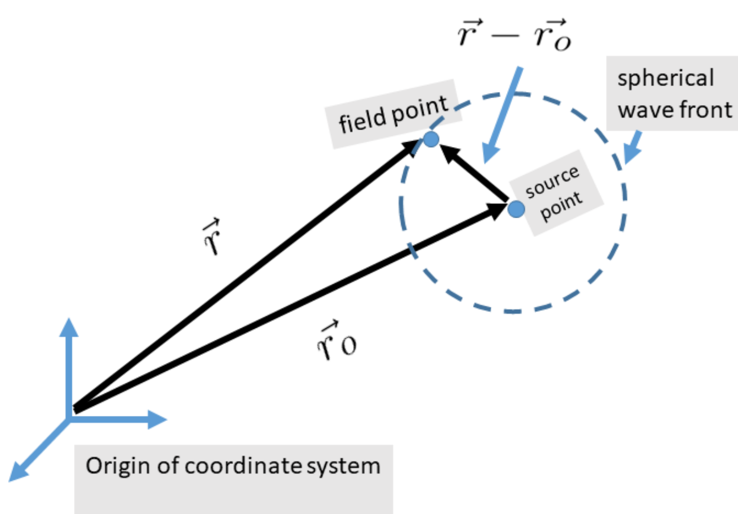

Figure 1: An acoustic source at the source point $\vec{r_0}$ producing the acoustic field at field point $\vec{r}$.

Equation (10) satisfies the inhomogeneous Helmholtz equation, for which the delta function on the RHS represents a point source of strength q at position $\vec{r_0}$ such that

$$(\nabla^2 + k^2)p = -q\delta(\vec{r}-\vec{r_0}) \tag{11}$$

Here are the key properties of the delta function $\delta(\vec{r}-\vec{r_0})$:
(1) $\delta(\vec{r}-\vec{r_0}) = 0$ for $\vec{r} \neq \vec{r_0}$
(2) $\int_V \delta(\vec{r}-\vec{r_0})dV = 1$
(3) $\int_V f(\vec{r})\delta(\vec{r}-\vec{r_0})dV = f(\vec{r_0})$ which is known as the "sifting property" of the delta function. See also Fahy (2001).

We further compress notation by defining $R = |\vec{r}-\vec{r_0}|$, such that

$$g = \frac{e^{ikR}}{4\pi R} \tag{12}$$

and call g the *free space* Green's function (Pierce 1989, Morse and Ingard, 1968) because g satisifies

$$(\nabla^2 + k^2)g = -\delta(\vec{r}-\vec{r_0}) \tag{13}$$

in an *unbounded* medium.

By *unbounded* medium we mean there are no nearby boundaries to reflect sound, and therefore the sound spreads uniformly away from the source while decaying in amplitude as $\sim 1/R$, where R is range from source.

A purely unbounded medium might be represented by two people having a conversation– each in their separate helium balloons far above land. But approximately unbounded media are everywhere. An excellent one you might experience this winter is being on snow and listening to sounds or speaking with someone nearby– the air above is unbounded and the snow below is highly absorptive of sound, hence sound reflection from the snow boundary is very weak. The opposite effect is experienced by having a conversation inside a stairwell where there are multiple echoes from the nearby reflective walls.

Note the physical dimension of g is $1/L$. As currently constructed, g embodies all the range-dependent and phase properties of a sound field with point source located at $\vec{r_0}$, but to bring a more useful dimension of pressure, g must be multiplied by some calibration constant.

To summarize, the function g given here represents a sound source (to within a calibration constant) that is concentrated at a point in the manner of a delta function in space, and g is as a solution is known as the *Green's function* for the problem at hand (Frisk, 1994). A formal proof of this solution is given at the end of these notes. This solution can either be a *harmonic* or an *impulsive* Green's function, depending on the time function characteristic of the source, $e^{\pm i\omega t}$ or $\delta(t)$. A Green's function concentrated in space and impulsive time is discussed in Pierce (1989), see also Tolstoy (1973). In this course we use primarily harmonic Green's function solutions, representing a single-frequency, or narrow band condition, and by Fourier superposition we can combine multiple frequencies to obtain a pulse of time duration τ and bandwidth $\sim 1/\tau$.

Finally, notice that since $|\vec{r}-\vec{r_0}| = |\vec{r_0}-\vec{r}|$ then one can exchange the field point and the source point with the result unchanged. This important property is call *reciprocity*, and the reciprocity principal is often exploited for calibrating microphones and hydrophones (Kinsler *et al.*, 1982). Furthermore, we no longer need to stick with spherically symmetric coordinates. For example, $\vec{r}$ and $\vec{r_0}$ are easily identified in Cartesian coordinates, as in $\vec{r} = [x, y, z]$ and $\vec{r_0} = [x_0, y_0, z_0]$.

We'll have opportunity to discusse the effect of boundaries, or boundary conditions, in later lectures. For example a major boundary condition to address with a sound source underwater is presence of sea surface and seabed boundary.

Acoustically compact source

Following the exercise concerning the $ka << 1$ we arrive an extraordinarily useful rule: if the characteristic scale L of source is such that $L << \lambda$ where λ is the acoustic wavelength, then the source is *acoustically compact*. Once the source is deemed acoustically compact the scale L is no longer relevant.

The source can be modeling as Eq. (9) or (10 where the source strength, q is determined empirically by measurement. For example, if p_{rms} is measured at range R m from the source, then we can estimate $|q|$ as follows

$$\frac{|q|}{4\pi}\frac{1}{\sqrt{2}}\frac{1}{R} = p_{rms} \tag{14}$$

giving at least a value for $|q|$. Often that is all we need anyway, as the real physics relating to sound propagation is embodied in Green's function g.

Lecture 8 Appendix: Formal proof of the Green's function g being a solution to the Helmholtz equation for a point source of sound

Let us next prove that g satisfies Eq.(13). First put the point source location $\vec{r_0}$ at the center of a coordinate system with no loss of generality. Then examine $g = \frac{e^{ikr}}{4\pi r}$ as a solution to

$$\nabla^2 g + k^2 g = -\delta(r) \tag{15}$$

where r is now an ordinary radial coordinate from the origin and there is no need to vectorize.

Now consider a volume V that does not include the origin; under these circumstances we have $\nabla^2 g + k^2 g = 0$ in view of the properties of the delta function given in these notes. The fact that g, a spherically symmetric wave so defined, is a solution to this *homogeneous* Helmholtz equation is already a settled issue. For example one can put $G = rg$ and G will be a plane wave solution as demonstrated previously.

Next we show that

$$\nabla^2 \frac{e^{ikr}}{r} + k^2 \frac{e^{ikr}}{r} = -4\pi\delta(r) \tag{16}$$

over a small volume V that now encloses the source at the origin. We set this up as follows:

$$\int_V \nabla^2 \frac{e^{ikr}}{r} dV + k^2 \int_V \frac{e^{ikr}}{r} dV = -4\pi \tag{17}$$

where the -4π again emerges from the basic property of the delta function.

We examine the two volume integrals separately, put the first equal to I_1 and the second equal to I_2. For I_1 use the divergence theorem to convert the I_1 volume integral into a surface integral giving

$$I_1 = \int_{A_\epsilon} \vec{n} \cdot \nabla \frac{e^{ikr}}{r} dA \tag{18}$$

where A_ϵ is area of a "very small" sphere that encloses the source point. Carefully lay out this surface integral as

$$I_1 = \int_0^{2\pi} d\phi \int_0^{\pi} [\frac{d}{dr}\frac{e^{ikr}}{r}]\epsilon^2 \sin\theta d\theta \tag{19}$$

the factor $[\frac{d}{dr}\frac{e^{ikr}}{r}]$ is evaluated at $r = \epsilon$, and observe that this will be $\frac{ik\epsilon - 1}{\epsilon^2}$. Thus in the limit of $\epsilon \to 0$ we find $I_1 = -4\pi$.

For I_2, recognize that dV equals $d\phi\epsilon^2 \sin\theta d\theta$ and thus this integral will equal 0 as $\epsilon \to 0$. Therefore, Eq. (17) is satisfied.

References

L. E. Kinsler, A. R. Frey, A. B. Coppens, and J. V. Sanders, *Fundamentals of Acoustics*, (John Wiley & Sons, New York, 1982)
G. V. Frisk,*Ocean and Seabed Acoustics* (Prentice Hall, Upper Saddle River, N.J., 1994)
D. T. Blackstock, *Fundamentals of Physical Acoustics* (Wiley-Interscience, New York, 2000).
Tolstoy, I. *Wave Propagation*, (McGraw-Hill International Series in the Earth and Planetary Sciences, New York, 1973)
Pierce, A. B, *Acoustics, An Introduction to its Physical Principals and Applications*, (Acoustical Society of America, and American Institute of Physics, 1989)
F. Fahy, *Foundations of Engineering Acoustics* (Elsevier Academic Press, San Diego, CA, 2001)
Morse P. and K. U. Ingard,*Theoretical Acoustics*, (Princeton University Press, Princeton, N.J., 1986)
Temkin, S. *Elements of Acoustics*,(Acoustical Society of America, and American Institute of Physics, 2001)